Connections between discriminants and the root distribution of polynomials with rational generating function

Khang Tran
Department of Mathematics and Computer Science
Truman State University

Abstract

Let $H_m(z)$ be a sequence of polynomials whose generating function $\sum_{m=0}^{\infty} H_m(z)t^m$ is the reciprocal of a bivariate polynomial $D(t, z)$. We show that in the three cases $D(t, z) = 1 + B(z)t + A(z)t^2$, $D(t, z) = 1 + B(z)t + A(z)t^3$ and $D(t, z) = 1 + B(z)t + A(z)t^4$, where $A(z)$ and $B(z)$ are any polynomials in $z$ with complex coefficients, the roots of $H_m(z)$ lie on a portion of a real algebraic curve whose equation is explicitly given. The proofs involve the $q$-analogue of the discriminant, a concept introduced by Mourad Ismail.

1 Introduction

In this paper we study the root distribution of a sequence of polynomials satisfying one of the following three-term recurrences:

\begin{align*}
H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-2}(z) &= 0, \\
H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-3}(z) &= 0, \\
H_m(z) + B(z)H_{m-1}(z) + A(z)H_{m-4}(z) &= 0,
\end{align*}

with certain initial conditions and $A(z), B(z)$ polynomials in $z$ with complex coefficients. For the study of the root distribution of other sequences of polynomials that satisfy three-term recurrences, see [8] and [10]. In particular, we choose the initial conditions so that the generating function is

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{D(t, z)}$$

where $D(t, z) = 1 + B(z)t + A(z)t^2$, $D(t, z) = 1 + B(z)t + A(z)t^3$, or $D(t, z) = 1 + B(z)t + A(z)t^4$. We notice that the root distribution of $H_m(z)$ will be the same if we replace 1 in the numerator by any monomial $N(t, z)$. If $N(t, z)$ is not a monomial, the root distribution will be different. The quadratic case $D(t, z) = 1 + B(z)t + A(z)t^2$ is not difficult and it is also mentioned in [13]. We

The author acknowledges support from NSF grant DMS-0838434 "EMSW21MCTP: Research Experience for Graduate Students" from the University of Illinois at Urbana-Champaign.
present this case in Section 2 because it gives some directions to our main cases, the cubic and quartic denominators $D(t, z)$, in Sections 3 and 4.

Our approach uses the concept of $q$-analogue of the discriminant ($q$-discriminant) introduced by Ismail [12]. The $q$-discriminant of a polynomial $P_n(x)$ of degree $n$ and leading coefficient $p$ is

$$\text{Disc}_q(P; q) = p^{2n-2}q^{n(n-1)/2} \prod_{1 \leq i < j \leq n} (q^{-1/2}x_i - q^{1/2}x_j)(q^{1/2}x_i - q^{-1/2}x_j)$$  \hspace{1cm} (1)$$

where $x_i$, $1 \leq i \leq n$, are the roots of $P_n(x)$. This $q$-discriminant is 0 if and only if a quotient of roots $x_i/x_j$ equals $q$. As $q \to 1$, this $q$-discriminant becomes the ordinary discriminant which is denoted by $\text{Disc}_2 P(x)$. For the study of resultants and ordinary discriminants and their various formulas, see [1], [2], [9], and [11].

We will see that the concept of $q$-discriminant is useful in proving connections between the root distribution of a sequence of polynomials $H_m(z)$ and the discriminant of the denominator of its generating function $D(t, z)$. We will show in the three cases mentioned above that the roots of $H_m(z)$ lie on a portion of a real algebraic curve (see Theorem 1, Theorem 3, and Theorem 5). For the study of sequences of polynomials whose roots approach fixed curves, see [5, 6, 7]. Other studies of the limits of zeros of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in $z$ are given in [3, 4]. The $q$-discriminant will appear as the quotient $q$ of roots in $t$ of $D(t, z)$. One advantage of looking at the quotients of roots is that, at least in the three cases above, although the roots of $H_m(z)$ lie on a curve depending on $A(z)$ and $B(z)$, the quotients of roots $t = t(z)$ of $D(t, z)$ lie on a fixed curve independent of these two polynomials. We will show that this independent curve is the unit circle in the quadratic case and two peculiar curves (see Figures 1 and 2 in Sections 3 and 4) in the cubic and quartic cases. From computer experiments, this curve looks more complicated in the quintic case $D(z, t) = 1 + B(z)t + A(z)t^5$ (see Figure 3 in Section 4).

As an application of these theorems, we will consider an example where $D(t, z) = 1 + (z^2 - 2z + a)t + tz^2$ and $a \in \mathbb{R}$. We will see that the roots of $H_m(z)$ lie either on portions of the circle of radius $\sqrt{a}$ or real intervals depending on the value $a$ compared to the critical values 0 and 4. Also, the endpoints of the curves where the roots of $H_m(z)$ lie are roots of $\text{Disc}_1 D(t, z)$. Interestingly, the critical values 0 and 4 are roots of the double discriminant $\text{Disc}_2 \text{Disc}_1 D(t, z) = 4096a^2(a - 4)$.

## 2 The quadratic denominator

In this section, we will consider the root distribution of $H_m(z)$ when the denominator of the generating function is $D(t, z) = 1 + B(z)t + A(z)t^2$.

**Theorem 1** Let $H_m(z)$ be a sequence of polynomials whose generating function is

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^2}$$

where $A(z)$ and $B(z)$ are polynomials in $z$ with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve $C_2$ defined by

$$\Im \left( \frac{B^2(z)}{A(z)} \right) = 0 \quad \text{and} \quad 0 \leq \Re \left( \frac{B^2(z)}{A(z)} \right) \leq 4,$$

and are dense there as $m \to \infty$. 


Proof Suppose $z_0$ is a root of $H_m(z)$ which satisfies $A(z_0) \neq 0$. Let $t_1 = t_1(z_0)$ and $t_2 = t_2(z_0)$ be the roots of $D(t, z_0)$. If $t_1 = t_2$ then $\text{Disc}_t D(t, z_0) = B^2(z_0) - 4A(z_0) = 0$. In this case $z_0$ belongs to $C_2$, and we only need to consider the case $t_1 \neq t_2$. By partial fractions, we have

$$
\frac{1}{D(t, z_0)} = \frac{1}{A(z_0)(t - t_1)(t - t_2)}
= \frac{1}{A(z_0)(t_1 - t_2)} \left( \frac{1}{t - t_1} - \frac{1}{t - t_2} \right)
= \frac{1}{A(z_0)} \sum_{m=0}^{\infty} \frac{t_1^{m+1} - t_2^{m+1}}{t_1 - t_2} t^{m+n}.
$$

(2)

Thus if we let $t_1 = qt_2$ then $q$ is an $(m+1)$-st root of unity and $q \neq 1$. By the definition of $q$-discriminant in (1), $q$ is a root of $\text{Disc}_t D(t, z_0); q)$ which equals

$$
q \left( B^2(z_0) - (q + q^{-1} + 2)A(z_0) \right).
$$

This implies that

$$
\frac{B^2(z_0)}{A(z_0)} = q + q^{-1} + 2.
$$

Thus $z_0 \in C_2$ since $q$ is an $(m+1)$-th root of unity.

The map $B^2(z)/A(z)$ maps an open neighborhood $U$ of a point on $C_2$ onto an open set which contains a point $2Rq + 2$, where $q$ is an $(m+1)$-th root of unity, when $m$ is large. From (2), there is a solution of $H_m(z)$ in $U$. The density of the roots of $H_m(z)$ follows.

Example We consider an example in which the generating function of $H_m(z)$ is given by

$$
\frac{1}{z^2t^2 + (z^2 - 2z + a)t + 1} = \sum_{m=0}^{\infty} H_m(z)t^m
$$

where $a \in \mathbb{R}$. Let $z = x + iy$. We exhibit the three possible cases for the root distribution of $H_m(z)$ depending on $a$:

1. If $a \leq 0$, the roots of $H_m(z)$ lie on the two real intervals defined by

$$(x^2 + a)(x^2 - 4x + a) \leq 0.$$

2. If $0 < a \leq 4$, the roots of $H_m(z)$ can lie either on the half circle $x^2 + y^2 = a$, $x \geq 0$, or on the real interval defined by $x^2 - 4x + a \leq 0$.

3. If $a > 4$, the roots of $H_m(z)$ lie on two parts of the circle $x^2 + y^2 = a$ restricted by $0 \leq x \leq 2$.

Indeed, by complex expansion, we have

$$
\Im \frac{B^2(z)}{A(z)} = \frac{2y(x^2 + y^2 - a)P}{(x^2 + y^2)^2} \quad \text{and} \quad \Re \frac{B^2(z)}{A(z)} = \frac{P^2 - Q^2}{(x^2 + y^2)^2},
$$

where

$$
P = ax - 2x^3 + x^3 - 2y^2 + xy^2 \quad \text{and} \quad Q = y(x^2 + y^2 - a).
$$
Theorem 1 yields three cases: \( y = 0, x^2 + y^2 - a = 0 \) or \( P = 0 \). Since \( \Re(B^2(z)/A(z)) \geq 0 \), all these cases give \( Q = 0 \). We note that if \( x^2 + y^2 - a = 0 \) then the condition \( \Re(B^2(z)/A(z)) \leq 4 \) reduces to
\[
x(a + x^2 + y^2)(ax - 4x^2 + x^3 - 4y^2 + xy^2) = 4a^2x(x - 2) \leq 0.
\]
Suppose \( a \leq 0 \). Then the condition \( Q = 0 \) implies that the roots of \( H_m(z) \) are real. The condition \( \Re(B^2(z)/A(z)) \leq 4 \) becomes
\[
(x^3 - 2x^2 + ax)^2 - 4x^4 = x^2(x^2 + a)(x^2 - 4x + a) \leq 0.
\]
Suppose \( 0 < a \leq 4 \). The roots of \( H_m(z) \) lie either on the half circle \( x^2 + y^2 - a = 0, x \geq 0 \) (from the inequality (3)), or on the real interval given by \( x^2 - 4x + a \leq 0 \) (from the inequality (4)). If \( a > 4 \) then the roots of \( H_m(z) \) lie on the two parts of the circle \( x^2 + y^2 - a = 0 \) restricted by \( 0 \leq x \leq 2 \) (from the inequality (3)).

We notice that in this example, the inequality \( \Re(B^2(z)/A(z)) \leq 4 \) gives the endpoints of the curves where the roots of \( H_m(z) \) lie. Thus, these endpoints are roots of \( \text{Disc}_t(1 + B(z)t + A(z)t^2) = B^2(z) - 4A(z) \). Moreover the critical values of \( a \), which are 0 and 4, are roots of the double discriminant of the denominator
\[
\text{Disc}_t \text{Disc}_t(1 + (z^2 - 2z + a)t + z^2t^2) = 4096a^3(a - 4).
\]
This comes from the fact that the endpoints of the fixed curves containing the roots of \( H_m(z) \) are the roots of \( \text{Disc}_t(1 + (z^2 - 2z + a)t + z^2t^2) \). When this discriminant has a double root as a polynomial in \( z \), some two endpoints of the fixed curves coincide. That explains the change in the shape of the root distribution.

3 The cubic denominator

In this section we show that in the cubic case \( D(t, z) = 1 + B(z)t + A(z)t^2 \), the roots of \( H_m(z) \) lie on a portion of a real algebraic curve. As we see in the proof of Theorem 1, we can first consider the distribution of the quotients of roots \( q = t_i/t_j \) of \( D(t, z) \), and then we can relate to the root distribution of \( H_m(z) \) using the \( q \)-discriminant. While in the previous section this quotient lies on the unit circle, in this section we show that this quotient lie on the curve in Figure 1.

Lemma 2 Suppose \( \zeta_1, \zeta_2 \neq 0 \) are complex numbers such that \( 1/\zeta_1 + 1/\zeta_2 + 1 = 0 \) and
\[
\frac{\zeta_1^{m+1}}{\zeta_1 - 1} = \frac{\zeta_2^{m+1}}{\zeta_2 - 1}.
\]
Then \( \zeta_1 \) and \( \zeta_2 \) lie on the union \( C_1 \cup C_2 \cup C_3 \) where the Cartesian equations of \( C_1, C_2 \) and \( C_3 \) are given by
\[
C_1 : \quad (x + 1)^2 + y^2 = 1, x \leq -\frac{1}{2},
\]
\[
C_2 : \quad x = -\frac{1}{2}, \quad -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2},
\]
\[
C_3 : \quad x^2 + y^2 = 1, x \geq -\frac{1}{2},
\]
and are dense there as \( m \to \infty \).
Figure 1: Distribution of the quotients of the roots of the cubic denominator

**Proof** We can rewrite (5) as

\[
\sum_{k=0}^{m} \zeta_1^k = \sum_{k=0}^{m} \zeta_2^k
\]

where we can replace \( \zeta_2 \) by \( -\zeta_1/(\zeta_1 + 1) \). By multiplying both sides by \((\zeta_1 + 1)^m\), we note that there are at most \(2m - 2\) solutions \( \zeta = \zeta_1 \neq 0, -2 \) counting multiplicity. Let \( m = 3n + k \) where \( k = 1, 2, 3 \). From implicit differentiation, we can check that the equation (5) has roots at \( e^{2\pi i/3}, e^{4\pi i/3} \) with multiplicity \( k - 1 \). After subtracting this number of roots from \(2m - 2\), we conclude that there are at most \(6n\) roots \( \zeta \neq 0, -1 \), \(e^{2\pi i/3}, e^{4\pi i/3} \). We first show that if \( \zeta \neq -2 \) is a root, then so is \( -\zeta - 1 \).

From the two equations in the hypothesis, we note that \( \zeta \neq 0, -1 \) and

\[
\sum_{k=0}^{m} \zeta^k = \sum_{k=0}^{m} \left( -\frac{\zeta}{\zeta + 1} \right)^k.
\]

Subtracting 1, then dividing by \( \zeta \) and multiplying both sides by \((\zeta + 1)^m\), we obtain

\[
0 = \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^m + \sum_{k=0}^{m-1} (\zeta + 1)^{m-k-1} (-\zeta)^k
\]

\[
= \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^{m-k-1} \left( (\zeta + 1)^{k+1} - (-1)^{k+1} \right)
\]

\[
= (\zeta + 2) \sum_{k=0}^{m-1} \zeta^k (\zeta + 1)^{m-k-1} \sum_{i=0}^{k} (\zeta + 1)^{k-i} (-1)^i
\]

\[
= (\zeta + 2) \sum_{k=0}^{m-1} \sum_{i=0}^{k} \zeta^k (-\zeta - 1)^{m-1-i}.
\]
By interchanging the summation and reversing the index of summation we obtain
\[
\sum_{k=0}^{m-1} \sum_{i=0}^{m-1} \zeta^k (-\zeta - 1)^{m-1-i} = \sum_{i=0}^{m-1} \sum_{k=i}^{m-1} \zeta^k (-\zeta - 1)^{m-1-i} = \sum_{i=0}^{m-1} \sum_{k=0}^{i} \zeta^{m-1-k} (-\zeta - 1)^{i}.
\]
Hence we have symmetry between \(\zeta\) and \(-\zeta - 1\) in the two double summations.

Our goal is to show that the number of roots \(\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}\) on \(C_1 \cup C_2 \cup C_3\) is at least \(6\alpha\), counting multiplicities. Then all roots will lie on \(C_1 \cup C_2 \cup C_3\) since we have at most \(6\alpha\) roots \(\zeta \neq 0, -2, e^{2\pi i/3}, e^{4\pi i/3}\). By the symmetry of roots mentioned above, if \(\zeta \neq -2\) is a solution in \(C_1\), then \((-2 - \zeta, -2 - \zeta)\) is a solution in \(C_2 \times C_3\). Hence there is a bijection between roots in \(C_1 \setminus \{-2\}\), \(C_2\) and \(C_3\). Thus if \(C_1 \setminus \{e^{2\pi i/3}, e^{4\pi i/3}\}\) contains at least \(2\alpha + 1\) roots then all of the roots lie on \(C_1 \cup C_2 \cup C_3\). Let \(\zeta = \zeta_1\) be a root on \(C_1 \setminus \{e^{2\pi i/3}, e^{4\pi i/3}\}\). Then the equation \(1/\zeta_1 + 1/\zeta_2 + 1 = 0\) gives \(\zeta_2 = \zeta\). Thus (5) gives
\[
\Im \frac{\zeta^{m+1} - 1}{\zeta - 1} = 0.
\]
Write \(\zeta = re^{i\theta}\) where \(r = -2\cos\theta\), \(\cos\theta \leq -1/2\). Then complex expansion yields
\[
r^{m+2}\sin m\theta - r^{m+1}\sin(m+1)\theta + r\sin \theta = 0.
\]
Divide \(r\), replace \(r\) by \(-2\cos\theta\) and combine the first two terms to obtain
\[
0 = (-1)^{m+1}\cos^m \theta (2\sin m\theta \cos \theta + \sin(m+1)\theta) + \sin \theta
\]
\[
= (-1)^{m+1}\cos^m \theta (2\sin(m+1)\theta - 2 \cos m\theta \sin \theta + \sin(m+1)\theta) + \sin \theta
\]
\[
= (-1)^{m+1}\cos^m \theta (2\sin(m+1)\theta + \sin m\theta \cos \theta - \cos m\theta \sin \theta) + \sin \theta
\]
\[
= (-1)^{m+1}\cos^m \theta (2\sin(m+1)\theta + \sin(m-1)\theta) + \sin \theta.
\]
We note that the right side has different signs if \(\sin(m+1)\theta = 1\) and \(\sin(m+1)\theta = -1\). Thus we can apply the Intermediate Value Theorem on several intervals whose boundaries are the solutions of \(\sin(m+1)\theta = \pm 1\). The equations \(\sin(m+1)\theta = \pm 1\) give
\[
(m+1)\theta = \pm \frac{\pi}{2} + 2j\pi.
\]
The condition \(2\pi/3 < \theta < 4\pi/3\) and the fact that \(m = 3n + k, k = 1, 2, 3\), yield
\[
n + \frac{k+1}{3} \pm \frac{1}{4} < j < 2n + \frac{2(k+1)}{3} \pm \frac{1}{4}.
\]
If \(k = 1\), we have at least \(2n + 1\) roots coming from \(2n + 1\) intervals formed by the \(2n + 2\) points
\[
\frac{2j\pi \pm \pi/2}{m+1},
\]
where \(n < j \leq 2n + 1\). If \(k = 2\), we have at least \(2n + 1\) roots coming from \(2n + 1\) intervals formed by the \(2n + 2\) points
\[
\left\{ \frac{2j - \pi/2}{m+1} : n+1 \leq j < 2n+2 \right\} \cup \left\{ \frac{2j + \pi/2}{m+1} : n+1 < j \leq 2n+2 \right\}.
\]
If $k = 3$, we have at least $2n + 1$ roots coming from $2n + 1$ intervals formed by the $2n + 2$ points

\[ \frac{2j\pi \pm \pi/2}{m + 1}, \]

where $n + 1 < j < 2n + 2$. The density follows from the distribution of $2n + 1$ roots mentioned above. The lemma follows.

**Theorem 3** Let $H_m(z)$ be a sequence of polynomials whose generating function is

\[ \sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^3} \]

where $A(z)$ and $B(z)$ are polynomials in $z$ with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve $C_3$ defined by

\[ \Im \frac{B^3(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq -\Re \frac{B^3(z)}{A(z)} \leq \frac{3^3}{2^2}, \]

and are dense there as $m \to \infty$.

**Proof** For a little simplification, we consider the roots of $H_{m-1}(z)$. Let $z_0$ be a root of $H_{m-1}(z)$ which satisfies $A(z_0) \neq 0$. Let $t_1 = t_1(z_0)$, $t_2 = t_2(z_0)$ and $t_3 = t_3(z_0)$ be the roots of $D(t, z_0) = 1 + B(z_0)t + A(z_0)t^3$. It suffices to consider $\text{Disc}_t(D(t, z_0)) = -4A(z_0)B^3(z_0) - 27A^2(z_0) \neq 0$. By partial fractions, the function $1/D(t, z_0)$ is

\[ \frac{1}{A(z_0)(t_1 - t_2)(t_1 - t_3)(t - t_1)} + \frac{1}{A(z_0)(t_2 - t_1)(t_2 - t_3)(t - t_2)} + \frac{1}{A(z_0)(t_3 - t_1)(t_3 - t_2)(t - t_3)}. \]

We expand $1/(t - t_i)$ using geometric series and write the expression above as

\[ \sum_{m=1}^\infty \frac{t_1^{m+1}t_2^m - t_1^m t_2^{m+1} - t_1^m t_3^{m+1} + t_2^m t_3^{m+1}}{A(z_0)t_1^m t_2^m t_3^m (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)} t^{m-1}. \]

Since $z_0$ is a root of $H_{m-1}(z)$, we have

\[ t_1^{m+1}t_2 - t_1^m t_2^{m+1} - t_1^m t_3^{m+1} + t_2^m t_3^{m+1} + t_1^m t_3^{m+1} - t_2^m t_3^{m+1} = 0. \]

We divide this equation by $t_3^{2m+1}$ and let $q = q_1 = t_1/t_3$, $q_2 = t_2/t_3$ to obtain

\[ q_1^{m+1}q_2^m - q_1^m q_2^{m+1} - q_1^m + q_2^{m+1} + q_1^m - q_2^m = 0 \]

where $q_1 + q_2 + 1 = 0$ since $t_1 + t_2 + t_3 = 0$. The equation can be written as

\[ q_1^m q_2^m (q_1 - q_2) - q_1^m (q_1 - 1) + q_2^m (q_2 - 1) = 0. \]

Since $q_1^m q_2^m (q_1 - q_2) = q_1^m q_2^m (q_1 - 1) - q_1^m q_2^m (q_2 - 1)$ and $q_1, q_2 \neq 0, 1$, this equation becomes

\[ \frac{q_1^m - 1}{q_1^m (q_1 - 1)} = \frac{q_2^m - 1}{q_2^m (q_2 - 1)}. \]
Let ζ₁ = 1/q₁ and ζ₂ = 1/q₂ and add 1 to both sides. Then

\[ \frac{ζ₁^{m+1} - 1}{ζ₁ - 1} = \frac{ζ₂^{m+1} - 1}{ζ₂ - 1}. \]

Thus ζ₁ and ζ₂ (and also q₁ and q₂) lie on the curve given in Lemma 2. Since q₁ and q₂ are given by quotients of two roots, they are roots of the q-discriminant given by

\[ \text{Disc}_t(D(t, z₀); q) = B₃(z₀)A(z₀)q²(1 + q)^2 - A²(z₀)(1 + q + q²)^3. \]

This gives

\[ \frac{B₃(z₀)}{A(z₀)} = \frac{(1 + q + q²)^3}{q²(1 + q)^2}. \]

It remains to show that the map

\[ f(q) = \frac{(1 + q + q²)^3}{q²(1 + q)^2} \]

maps the curve in Figure 1 to the real interval \([-27/4, 0]\). Let q be a point on the this curve. We note that

\[ f(q) = f(-1 - q) = -\frac{(q^{-1} + 1 + q)^3}{q^{-1} + 2 + q}. \]

Since q lies on the curve in Figure 1, we have the three possible cases \( \bar{q} = -1 - q \), \(|q| = 1 \) or \(|-1 - q| = 1 \). In the first case, \( \Im f(q) = 0 \) since \( f(q) = \bar{f}(q) \). In the second and third cases, \( \Im f(q) = 0 \) since \( q + q^{-1} \in \mathbb{R} \) and \( f(q) = f(-1 - q) \). Furthermore, \( f(q) \) attains its minimum and maximum when \( q = 1 \) and \( q = e^{2πi/3} \) respectively. The density of the roots of \( Hₘ(z) \) follows from similar arguments as in the proof of Theorem 1.

### 4 The quartic denominator

In this section, we will show that in the case \( D(t, z) = 1 + B(z) + A(z)t^4 \) the roots of \( Hₘ(z) \) lie on a portion of a real algebraic curve. Similar to the approach in the previous sections, we first consider the distribution of the quotients of roots of \( D(t, z) \). Before looking at these quotients, let us recall that the Chebyshev polynomial of the second kind \( Uₘ(z) \) is

\[ Uₘ(z) = \frac{\sin(m + 1)θ}{\sin θ} \]

where

\[ z = \cos θ. \]

Suppose \( z₁, z₂ \in \mathbb{C} \) such that \( \mid z₁ \mid = \mid z₂ \mid \). Let \( e^{2iθ} = z₁/z₂ \) and \( z = \cos θ \). If \( k \) is a positive integer then

\[
\frac{z₁^k - z₂^k}{z₁ - z₂} = (z₁z₂)^{(k-1)/2}(z₁/z₂)^{m/2} - (z₂/z₁)^{m/2} = (z₁z₂)^{(k-1)/2}Uₘ(z).
\]
By analytic continuation, we can extend this identity to any pair of complex numbers $z_1$ and $z_2$ with

$$2z = \left( \frac{z_1}{z_2} \right)^{1/2} + \left( \frac{z_2}{z_1} \right)^{1/2}.$$  

**Lemma 4** Suppose $z_0$ is a root of $H_m(z)$ and $q = q(z_0)$ is a quotient of two roots in $t$ of $1 + B(z_0)t + A(z_0)t^4$. Then the set of all such quotients belongs to the curve depicted in Figure 2, where the Cartesian equation of the quartic curve on the left is

$$1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2y^2 + y^4 = 0,$$

and the curve on the right is the unit circle with real part at least $-1/3$. All such quotients are dense on this curve as $m \to \infty$.

![Figure 2: Distribution of the quotients of the roots of the quartic denominator](image)

**Proof** For each $z_0 \in \mathbb{C}$, let $t_1 = t_1(z_0)$, $t_2 = t_2(z_0)$, $t_3 = t_3(z_0)$, and $t_4 = t_4(z_0)$ be the roots of the denominator $1 + B(z_0)t + A(z_0)t^4$. By partial fractions, we have

$$\frac{1}{1 + B(z_0)t + A(z_0)t^4} = \frac{1}{A(z_0)(t - t_1)(t - t_2)(t - t_3)(1 - t_4)} = \sum_{m=0}^{\infty} H_m(z_0)t^m,$$

where

$$A(z_0)H_m(z_0) = \frac{1}{t_1^{m+1}(t_1 - t_2)(t_1 - t_3)(t_1 - t_4)} + \frac{1}{t_2^{m+1}(t_2 - t_1)(t_2 - t_3)(t_2 - t_4)}$$

$$+ \frac{1}{t_3^{m+1}(t_3 - t_1)(t_3 - t_2)(t_3 - t_4)} + \frac{1}{t_4^{m+1}(t_4 - t_1)(t_4 - t_2)(t_4 - t_3)}.$$
Let \( q_1 = t_1/t_4, \ q_2 = t_2/t_4, \ q_3 = t_3/t_4 \). For a little reduction in the powers of \( q_i, 1 \leq i \leq 3 \), we will consider the roots of the polynomial \( H_{m-2}(z) \). We put all terms of \( A(z_0)H_{m-2}(z_0) \) over a common denominator and then divide the numerator by \( t_4^m \). The condition \( H_{m-2}(z_0) = 0 \) implies

\[
0 = q_1^{m+1}(-q_2^m q_3^{m-1} (q_2 - q_3) + q_2^m q_3^{m-1} - q_2^{m-1} q_3^m + q_2^{m-1} q_3^m - q_2^m q_3^{m-1})
+ q_1 q_2^m q_3^{m-1} (q_2 - q_3) - q_2^{m+1} q_3^{m-1} + q_2^{m+1} q_3^{m-1} - q_3^{m-1} q_2^m
+ q_2^{m-1} q_3^m (q_2 - q_3) - q_2^{m-1} q_3^m - q_2^{m-1} q_3^m + q_2^{m+1} q_3^{m-1} q_3^{m-1}.
\]

(7)

The fact

\[
t_1 + t_2 + t_3 + t_4 = 0
\]
\[
t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4 = 0
\]
gives

\[
q_2 + q_3 = -1 - q_1 \quad \text{and} \quad q_2 q_3 = q_1^2 + q_1 + 1.
\]

From the symmetric reductions, the right side of (7), after being divided by \( q_2 - q_3 \), is a polynomial in \( q_1 \) of degree \( 3m - 1 \). We used a computer algebra system to check for the root distribution of this polynomial in the case \( m \leq 5 \). We now assume that \( m \geq 6 \). We will show that the number of roots \( q_1 \) lying on the two curves in Figure 2 is at least \( 3m - 1 \). The first step is to show that if the set of \( q_1 \) belongs to the unit circle with \( \Re q_1 \geq -1/3 \) and is dense there as \( m \to \infty \) then the set of \( q_2 \) and \( q_3 \) belongs to the quartic curve given in the lemma and is dense on this quartic curve as \( m \to \infty \). Then we will find the number of roots \( q_1 \) on the unit circle with \( \Re q_1 \geq -1/3 \).

Suppose \( q_1 = e^{i \pi \theta} \) lies on the unit circle and \( 1 \geq \cos \theta \geq -1/3 \). We note that \( q_2 \) and \( q_3 \) are the two roots of the equation

\[
f(q) := q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0.
\]

Thus the quadratic formula gives

\[
q = \frac{-1 - e^{i \theta} \pm i e^{i \theta/2} \sqrt{6 \cos \theta + 2}}{2}.
\]

Splitting the real and imaginary parts of the function on the left side, we leave it to the reader to check that this function maps the interval \( 1 \geq \cos \theta \geq -1/3 \) to the quartic curve

\[
1 + 2x + 2x^2 + 2x^3 + x^4 - 2y^2 + 2xy^2 + 2x^2 y^2 + y^4 = 0.
\]

We now compute the number of roots \( q_1 = e^{i \pi \theta} \) with \( \cos \theta \geq -1/3 \). We first consider \( q_1 \neq \pm i, 1 \). Let

\[
2 \zeta = \left( \frac{q_2}{q_3} \right)^{1/2} + \left( \frac{q_3}{q_2} \right)^{1/2}.
\]

Equation (6) gives

\[
\frac{q_2^m - q_3^m}{q_2 - q_3} = (q_2 q_3)^{(m-1)/2} U_{m-1} (\zeta)
\]
where
\[ \zeta^2 = \frac{1}{4} \left( q_2 + q_3 \right)^2 = \frac{(q_1 + 1)^2}{4(q_1^2 + q_1 + 1)} = \frac{1}{4(2\cos\theta + 1)} + \frac{1}{4} \in \mathbb{R}. \]  
(8)

We divide (7) by \( q_2 - q_3 \) and rewrite it in terms of Chebyshev polynomials:
\[ 0 = U_m(\zeta)\left(-q_1^m + q_1^{m-1}(q_1^2 + q_1 + 1)^{m/2} + U_{m-1}(\zeta)(q_1^{m+1} - q_1^{m-1})(q_1^2 + q_1 + 1)^{(m-1)/2} + U_{m-2}(\zeta)(-q_1^{m+1} + q_1^m)(q_1^2 + q_1 + 1)^{(m-2)/2} + (q_1^2 + q_1 + 1)^{m-1}(-3q_1^{m+1} - 2q_1^m - q_1^{m-1} + q_1^2 + 2q_1 + 3) \right). \]

We divide this equation by \( q_1^{-3(m-2)/2}(1-q_1)q_1^{-1} + 1\) and write \( (q_1^m - 1)/(q_1 - 1) \) in terms of Chebyshev polynomials. We obtain
\[ 0 = U_m(\zeta) + 2\zeta U_{m-1}(\zeta) + U_{m-2}(\zeta)/(q_1 + q_1^{-1} + 1) + (q_1 + q_1^{-1} + 1)^{m/2-1} \left( 3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta) \right), \]
where
\[ \zeta^2 = \frac{(q_1 + 1)^2}{4q_1} = \frac{2\cos\theta + 2}{4}. \]  
(9)

Finally, from (8) we can replace \( 1/(q_1 + q_1^{-1} + 1) \) by \( 4\zeta^2 - 1 \) and use the recurrence definition of the Chebyshev polynomials to rewrite this equation in the symmetric form below:
\[ 0 = (4\zeta^2 - 1)^{(m-2)/4} \left( 3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta) \right) + (4\zeta^2 - 1)^{(m-2)/4} \left( 3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta) \right). \]  
(10)

From this symmetric form, the right expression remains the same if we interchange \( \zeta \) and \( \xi \) or if we interchange \( \cos\theta \) and \( -\cos\theta/(2\cos\theta + 1) \) (from (8) and (9)). Thus the numbers of roots \( q_1 \) are the same in the two cases \( 0 < \cos\theta < 1 \) and \(-1/3 < \cos\theta < 0\). It is sufficient to count the number of roots \( 0 < \cos\theta < 1 \) or \( 1/2 < \zeta^2 < 1 \). Let \( \cos\alpha = \xi \) and \( U_m(\xi) = \sin(m + 1)\alpha/\sin\alpha \) where \(-\pi/4 < \alpha < \pi/4\), \( \alpha \neq 0 \). The idea is to show that in this case the summand
\[ (4\zeta^2 - 1)^{(m-2)/4} \left( 3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta) \right) \]  
(11)
dominates the right expression of (10). Since \( \zeta^2 \) and \( \xi^2 \) in (10) are real numbers and the Chebyshev polynomials in this equation are either even or odd, we can apply the Intermediate Value Theorem. We note that (11) has different signs when \( \sin(m + 1)\alpha = 1 \) and when \( \sin(m + 1)\alpha = -1 \). Suppose \( \sin(m + 1)\alpha = \pm 1 \) and \(-\pi/4 < \alpha < \pi/4\). Since
\[ 4\zeta^2 - 1 = \frac{1}{1 + 2\cos\theta} < 1, \]
it suffices to show
\[ \left| (4\zeta^2 - 1)^{(m-2)/2} \left( 3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi) \right) \right| \geq |3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)|. \]  
(12)
Let \( \zeta = \cos \beta \). Using the fact that \( 1/3 < \zeta^2 < 1/2 \) and \( U_m(\zeta) = \sin(m + 1)\beta / \sin \beta \), we obtain the following upper bound for the right hand side of (12):

\[
|3U_m(\zeta) + 2U_{m-2}(\zeta) + U_{m-4}(\zeta)| \leq 6\sqrt{2}.
\]

Since

\[
\alpha = \frac{\pi}{4} \left( 4k \pm 2 \right), \quad m + 1
\]

where \( k \in \mathbb{Z} \) and \(-\pi/4 < \alpha < \pi/4\), we have

\[
|\alpha| \leq \frac{\pi}{4} \left( 1 - \frac{1}{m + 1} \right).
\]

Thus

\[
\cos \alpha \geq \frac{\sqrt{2}}{2} \left( \cos \frac{\pi}{4(m + 1)} + \sin \frac{\pi}{4(m + 1)} \right).
\]

This inequality and (9) give

\[
2 \cos \theta = 4 \cos^2 \alpha - 2 \geq 4 \sin \frac{\pi}{2(m + 1)}.
\]

From the definition of the Chebyshev polynomial, we have

\[
U_{m-2}(\xi) = \frac{\sin(m - 1)\alpha}{\sin \alpha} = \frac{\sin(m + 1)\alpha \cos 2\alpha - \cos(m + 1)\alpha \sin 2\alpha}{\sin \alpha} = \frac{\sin(m + 1)\alpha \cos 2\alpha}{\sin \alpha}.
\]

With similar computations for \( U_{m-4}(\xi) \), we obtain

\[
|3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)| = \frac{|\sin(m + 1)\alpha||3 + 2 \cos 2\alpha + \cos 4\alpha|}{|\sin \alpha|}.
\]

Since \( \sin(m + 1)\alpha = \pm 1 \) and \( \cos 2\alpha \geq 0 \), the right side is at least \( 2\sqrt{2} \). We combine this with (14) to have

\[
|2 \cos \theta + 1|^{(m-2)/2} |3U_m(\xi) + 2U_{m-2}(\xi) + U_{m-4}(\xi)| \geq \left( 1 + 4 \sin \frac{\pi}{2(m + 1)} \right)^{(m-2)/2} 2\sqrt{2} \geq 6\sqrt{2}
\]

when \( m \geq 6 \). The inequality (12) follows. By the Intermediate Value Theorem, we have at least one root when \( \sin(m + 1)\alpha \) changes between \(-1\) and \( 1 \) with \(-\pi/4 < \alpha < \pi/4\). From the formula (13), the number of roots \( q_1 \) when \( 0 < \cos \theta < 1 \) is at least \( 2([((m - 2)/4)) \). By symmetry, the number of roots \( q_1 \neq \pm i, 1 \) with \( \Re q_1 > -1/3 \) on the unit circle is at least \( 4([((m - 2)/4)) \). Note that each of these roots gives two more roots \( q_2 \) and \( q_3 \) on the quartic curve.
It remains to check the multiplicities of \( q_1 = \pm i, 1 \) in the equation (7). We note that this equation has a root \( q_1 = 1 \) with multiplicity at least 1. In the case \( q_1 = 1 \) we obtain four more roots \( q_2, q_2^{-1}, q_3, \) and \( q_3^{-1} \). We now consider the case \( q_1 = \pm i \). The equation \( q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0 \) where \( q = q_2, q_3 \) gives \( (q_2, q_3) = (-1, i) \) or \( (q_2, q_3) = (i, -1) \) when \( q_1 = i \) and \( (q_2, q_3) = (-1, -i) \) or \( (q_2, q_3) = (-i, 1) \) when \( q_1 = -i \). Hence each of the roots \( q_1 = \pm i \) gives us another root at \(-1\) with the same multiplicity. To check the multiplicities at \( q_1 = \pm i \), we need to differentiate the equation (7) with respect to \( q_1 \). We obtain its derivatives by applying implicit differentiation to the equation \( q^2 + (1 + q_1)q + q_1^2 + q_1 + 1 = 0 \). After substituting \( q_1 = \pm i \) in (7) and its derivatives, we see that the multiplicity of \( \pm i \) is

\[
\begin{align*}
2 & \quad \text{if } m = 4k \\
3 & \quad \text{if } m = 4k + 1 \\
0 & \quad \text{if } m = 4k + 2 \\
1 & \quad \text{if } m = 4k + 3
\end{align*}
\]

The table below tabulates the \( 3m - 1 \) roots of (7).

<table>
<thead>
<tr>
<th>( q_1 = e^{\alpha i} ) if ( q_1 &gt; -1/3 ), ( q_1 \neq \pm i, 1 )</th>
<th>( q_1 = 1 )</th>
<th>( q_1 = \pm i )</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 4k )</td>
<td>( m = 4k + 1 )</td>
<td>( m = 4k + 2 )</td>
<td>( m = 4k + 3 )</td>
</tr>
<tr>
<td>( 3(4k - 4) )</td>
<td>( 3(4k - 4) )</td>
<td>( 12k )</td>
<td>( 12k )</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>12k - 1</td>
<td>12k + 2</td>
<td>12k + 5</td>
<td>12k + 8</td>
</tr>
</tbody>
</table>

All the roots counted on the table lie on the curves given in the lemma. The number of roots counted equals the number of possible roots which is \( 3m - 1 \). Also, as a consequence of the Intermediate Value Theorem applied to the intervals formed by \( \sin(m + 1)\alpha = \pm 1 \), the roots \( q_1 \) are dense on the portion of the unit circle with real part at least \(-1/3\). The lemma follows.

**Theorem 5** Let \( H_m(z) \) be a sequence of polynomials whose generating function is

\[
\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^4}
\]

where \( A(z) \) and \( B(z) \) are polynomials in \( z \) with complex coefficients. The roots of \( H_m(z) \) which satisfy \( A(z) \neq 0 \) lie on the curve \( C_4 \) defined by

\[
\exists B^4(z)/A(z) = 0 \quad \text{and} \quad 0 \leq \Re B^4(z)/A(z) \leq \frac{4^4}{3^3},
\]

and are dense there as \( m \to \infty \).

**Proof** From the definition of \( q \)-discriminant in (1), we have

\[
\text{Disc}_t(1 + B(z)t + A(z)t^4; q) = -A^2(z)B^4(z)q^3(1 + q + q^2)^3 + A^3(z)(1 + q + q^2 + q^3)^4.
\]

If \( q \) is a quotient of two roots of \( 1 + B(z)t + A(z)t^4 \), then

\[
\frac{B^4(z)}{A(z)} = \frac{(1 + q + q^2 + q^3)^4}{q^3(1 + q + q^2)^3}.
\]
Let $f(q)$ be the function on the right side. We note that $f(q)$ maps $q_1 = e^{i\theta}$ with $\Re q_1 \geq -1/3$ to the real interval $[0, 4^{1/3}]$ since

$$f(q_1) = \frac{(q_1^{3/2} + q_1^{-3/2} + q_1^{1/2} + q_1^{-1/2})^4}{(q_1 + q_1^{-1} + 1)^3}.$$ 

If $q$ is a point on the quartic curve in Lemma 4 then $q$ and $q_1$ are related by

$$q_1^2 + q^2 + q_1 q + q_1 + q + 1 = 0.$$ 

Multiplying this equation by $q_1 - q$, we obtain

$$q_1^3 + q_1^2 + q_1 = q^3 + q^2 + q.$$ 

Thus by the definition of $f(q)$, we have $f(q) = f(q_1)$. Since

$$\text{Disc}_t(1 + B(z)t + A(z)t^4) = -3^3 A^2(z)B^4(z) + 4^4 A^3(z),$$

the roots of $H_m(z)$ lie on the curve $C_4$. The density of these roots follows from arguments similar to those in the proof of Theorem 1.

**Remark:** One may try to find the root distribution of $H_m(z)$ in the case $D(t, z) = 1 + B(z)t + A(z)t^5$. From computer experiments, the distribution of the quotients of roots of $D(t, z)$ in the case $m = 50$ is given in the figure below.

![Figure 3: Distribution of the quotients of roots of the quintic denominator](image)

We end this paper with the following conjecture.

**Conjecture 6** Let $H_m(z)$ be a sequence of polynomials whose generating function is

$$\sum H_m(z)t^m = \frac{1}{1 + B(z)t + A(z)t^4}$$
where $A(z)$ and $B(z)$ are polynomials in $z$ with complex coefficients. The roots of $H_m(z)$ which satisfy $A(z) \neq 0$ lie on the curve $C_n$ defined by

$$\Im \frac{B^n(z)}{A(z)} = 0 \quad \text{and} \quad 0 \leq (-1)^n \frac{\Re B^n(z)}{A(z)} \leq \frac{n^n}{(n-1)^{n-1}},$$

and are dense there as $m \to \infty$.

References


