Problems for lecture 12

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1. The definition of $\lim_{x\to c} g(x) = L$ says that for every $\epsilon > 0$, we can find $\delta > 0$ so that if $0 < |x - c| < \delta$ then
$$|g(x) - L| < \epsilon.$$
Use this definition to prove the Squeeze Theorem which says if we have $f(x) \le g(x) \le h(x)$, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ then $\lim_{x\to c} g(x) = L$. Hint: write the inequality $|g(x) - L| < \epsilon$ as $L - \epsilon < g(x) < L + \epsilon$.

2. We know that if we can find two sequences $x_n \to c$, $y_n \to c$, $x_n \ne c$, $y_n \ne c$ so that $\lim f(x_n) \ne \lim f(y_n)$ then
$$\lim_{x\to c} f(x)$$
does not exists. Use this fact to prove that the following limits do not exist.

   (a) $\lim_{x\to 0} |x|/x$

   (b) $\lim_{x\to 0} g(x)$ where
$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
   The function $g(x)$ is called the Dirichlet function. Recall that $\mathbb{Q}$ is the set of rational numbers of the forms a/b where a and b are two integers. Hint: you can use the fact $\sqrt{2}$ is not a rational number (it is not trivial but I hope you have seen the proof of this in Math 200).

   (c) $\lim_{x\to 0} \sin(1/x)$

3. We have showed that $\lim_{x\to c} f(x) = L$ is the same as the fact that for any sequence $x_n \to c$, $x_n \ne c\ \forall n$, one has $f(x_n) \to L$ (limit as a sequence of numbers). Also a function is continuous at c if $\lim_{x\to c} f(x) = f(c)$. We combine the two sentences above to say that $f(x)$ is continuous at c if and only if for any sequence $x_n \to c$, one has $f(x_n) \to f(c)$ (note that we remove the condition $x_n \ne c$ here as noted in the lecture). Use this fact for the following problem.
Suppose $f(x)$ is continuous at c and $g(x)$ is continuous at $f(c)$. Show that the composite function $g(f(x))$, if well-defined, is continuous at c. Hint: Let x_n be any sequence with $x_n \to c$. Show that $g(f(x_n)) \to g(f(c))$.