Problems for lecture 19

1. A function $f : A \to \mathbb{R}$ is called Lipschitz if there is a number $M > 0$ so that

$$\left| \frac{f(x) - f(y)}{x - y} \right| < M$$

for all $x, y \in A$, $x \neq y$.

   (a) Show that if $f : A \to \mathbb{R}$ is Lipschitz then it is uniformly continuous on A. Hint: Refer to the ϵ, δ definition of a uniformly continuous function.

   (b) Consider the function $f(x) = \sqrt{x}$ on $[0, 1]$. This function is uniformly continuous on $[0, 1]$ because it is continuous on a compact set. Show that this function is not Lipschitz. Hint: Show that for any big $M > 0$ the condition

$$\left| \frac{f(x) - f(y)}{x - y} \right| < M$$

   is not true when $x = 0$ and y is small. Be clear in your arguments.

2. Recall that one version of the Intermediate Value Theorem says if $f(x)$ is continuous on $[a, b]$ where $f(a) < 0$ and $f(b) > 0$ then there is a point $c \in (a, b)$ so that $f(c) = 0$. Use this version of the theorem to prove the following version of this theorem. Let $f : [a, b] \to R$ be a continuous function with $f(a) < f(b)$. Show that if L is a number satisfying $f(a) < L < f(b)$ then there exists a point $c \in (a, b)$ where $f(c) = L$.

3. Let f be a continuous function on the closed interval $[0, 1]$ so that $f([0, 1]) \subseteq [0, 1]$. Prove that there is a value $c \in [0, 1]$ such that $f(c) = c$. Hint: apply the Intermediate Value Theorem to the function $x - f(x)$.