Problems for lecture 20

1. Assume that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable at a point $c \in A$.

   (a) Suppose $g(c) \neq 0$. Use the fact that $g(x)$ is continuous at c to show that there is a $\delta > 0$ so that $g(x) \neq 0$ whenever $|x - c| < \delta$. Hint: you can choose $\epsilon = |g(c)|/2$ in the ϵ-definition of continuity.

   (b) Suppose $g(c) \neq 0$. Part (b) says that $f(x)/g(x)$ is defined on an open interval containing c. Prove that $f(x)/g(x)$ is differentiable at c and

   $$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

   Hint: Note that

   $$\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)} = \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}$$

   and apply the same trick that we use for the derivative of a product in the lecture.

2. Consider the function

   $$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   (a) Show that $f(x)$ is continuous at 0.

   (b) Show that this function is not differentiable at 0.