

Problems for lecture 21

1. Recall that a function $f : A \rightarrow \mathbb{R}$ is Lipschitz if there is a number $M > 0$ so that

$$\left| \frac{f(y) - f(x)}{y - x} \right| < M$$

for all $x, y \in A$ $x \neq y$.

- (a) Consider the function $f(x) = \sqrt{x}$ on $(0, 1)$. In a previous homework, we showed that this function is not Lipschitz on $(0, 1)$. Show that this function is differentiable at any point $c \in (0, 1)$ and $f'(c) = \frac{1}{2\sqrt{c}}$.
- (b) Consider the function $g(x) = |x|$ on $(-1, 1)$. In a lecture we show that $g(x)$ is not differentiable at 0. Show that $g(x)$ is Lipschitz on $(-1, 1)$.

2. Show that the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable at 0 and find $f'(0)$.