Problems for lecture 23

March 18, 2015

1. Let $f(x)$ be a differentiable function on a closed interval $[a, b]$ and $f'(a) < \alpha < f'(b)$. In the proof of the Darboux's theorem in the lecture today, we consider the function $g(x) = f(x) - \alpha x$ and say that $g(x)$ attains a minimum at a point $c \in [a, b]$ (i.e., $g(c) \leq g(x)$ for all $x \in [a, b]$). Show that $c \neq a$. Hint: follow the argument that $c \neq b$ in the lecture.

2. The proof the Darboux's Theorem in the lecture today (when we are showing $c \neq b$) may give us a FALSE impression that if $f'(c) > 0$ at a point c then the function is increasing in a small interval around c. We say that a function is increasing on $\mathbb{R}$ if $f(x) \leq f(y)$ whenever $x \leq y$. Consider the function

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

   (a) Show that this function is differentiable on $\mathbb{R}$ and $f'(0) > 0$.

   (b) Show that this function is NOT increasing on any open interval containing 0. Hint: Let $(-\delta, \delta)$ be any interval around 0. Try to find two sequences $x_n \to 0$, $y_n \to 0$ (so that $x_n, y_n \in (-\delta, \delta)$ when n is large) where $x_n \leq y_n$ but $f(x_n) > f(y_n)$ for large n. Try to find two sequences where the sine of one of them is 0 and the sine of the other is 1. This problem may require some computations.