Problems for lecture 20

March 25, 2015

1. We assume the theorem that if $f_n \rightrightarrows f$ and $f_n(x)$ is continuous on A for each n then $f(x)$ is continuous on A. Consider the sequence of functions

$$f_n(x) = \frac{x}{1+x^n}.$$

   (a) Find a function $f(x)$ so that $f_n \to f$ on $[0, \infty)$.

   (b) Prove by contradiction that $f_n(x)$ does not converge uniformly to $f(x)$ on $[0, \infty)$.

   (c) Prove that $f_n(x)$ converges uniformly to $f(x)$ on $[0, 1/2]$.

2. Consider the function

$$f_n(x) = \frac{nx}{1+nx^2}.$$

   (a) Find a function $f(x)$ where $f_n \to f$ for all $x \in (0, \infty)$.

   (b) Prove that $f_n(x)$ converges uniformly to $f(x)$ on $[1, \infty)$.

   (c) Note that if we can find a sequence $x_n \in A$ so that

$$\lim_{n\to\infty} |f_n(x_n) - f(x_n)| > 0$$

   then $f_n(x)$ does not converge uniformly to $f(x)$ on A. Use this idea to prove that $f_n(x)$ does not converge uniformly to $f(x)$ on $(0, \infty)$.

3. The Cauchy inequality says if a and b are two positive numbers then $a + b \geq 2\sqrt{ab}$. Show that the sequence of functions

$$f_n(x) = \frac{x}{1+nx^2}$$

   converges uniformly to the constant 0 function. Hint: when $x \neq 0$, we divide the numerator and the denominator of $|f_n(x)|$ by $|x|$ to obtain

$$\frac{|x|}{1+nx^2} = \frac{1}{\frac{1}{|x|} + n|x|}.$$

   Try to use the Cauchy inequality from here.