Problems for lecture 30

1. Consider the sequence of functions $f_n(x) = x + 1/n$ on $\mathbb{R}$.

   (a) Show that $f_n(x)$ converges uniformly to the function $f(x) = x$ on $\mathbb{R}$.

   (b) Show that $(f_n(x))^2$ does NOT converge uniformly to the function x^2 on $\mathbb{R}$. Hint: try to find a sequence x_n so that
   $$\lim |f_n^2(x_n) - f^2(x_n)| > 0.$$
   Do you see why this does not contradict with Problem 2 in Lecture 28? (when we we take $f_n(x) = g_n(x)$ in this problem).

2. Consider the function
   $$f_n(x) = \frac{x}{1 + nx^2}$$
   on any interval $[a, b]$. In a previous homework, we showed that f_n converges uniformly to $f(x) \equiv 0$.

   (a) Find $f_n'(x)$ and the function $g(x)$ where $f_n'(x) \to g(x)$. Do this part carefully.

   (b) Recall that uniform convergence of a sequence of function has to preserve continuity. Use this to show that the convergence in part (a) cannot be uniform.

   (c) Recall that if we have the uniform convergence of the sequence of derivatives and the convergence of the sequence of functions at one point, then the derivative of a limit has to be the same as the limit of derivatives. Use this to explain why the convergence in part (a) cannot be uniform.