Problems for lecture 31

1. Recall the Cauchy condition that a sequence of functions $h_n(x)$ is uniformly convergent on A if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that if $n, m \geq N$ then

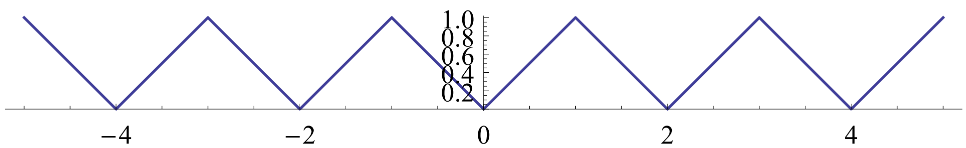
$$|h_n(x) - h_m(x)| < \epsilon \text{ for all } x \in A.$$

Prove that a series of functions $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly on A if and only if for every $\epsilon > 0$, we can find an N so that if $n > m \geq N$ then

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon$$

for all $x \in A$. Hint: apply the Cauchy condition to the sequence of functions of partial sums $s_m(x)$.

2. Consider the function $h(x) = |x|$ on $[-1, 1]$. We extend the domain $[-1, 1]$ to the real line by repeating this interval (i.e., by defining $h(x + 2) = h(x)$). See the graph below



Show that the series

$$\sum_{n=0}^{\infty} \frac{h(2^n x)}{2^n}$$

is uniformly convergence to a function $s(x)$. Although do not know the formula of $s(x)$, explain why $s(x)$ is a continuous function on $\mathbb{R}$.