Problems for lecture 32

April 10, 2015

1. Consider the series of functions $\sum_{k=0}^{\infty} x^k$.

   (a) Find a formula for the partial sum $s_m(x) = 1 + x + x^2 + x^3 + \cdots + x^m$
   $$s_m(x) = \frac{1 - x^{m+1}}{1 - x}$$
   and from this formula deduce that the series $\sum_{k=0}^{\infty} x^k$ converges pointwise to the function
   $$s(x) = \frac{1}{1 - x}$$
   on $(-1, 1)$ (I have mentioned this fact in the class, I want to prove it here). Hint: To prove the formula of $s_m(x)$, try to expand and simplify $(1 - x)s_m(x)$.

   (b) Show that the convergence of this series $\sum_{k=0}^{\infty} x^k$ to $s(x)$ is not uniform on $(-1, 1)$. Hint: try to find a sequence x_m so that
   $$\lim |s_m(x_m) - s(x_m)| > 0$$
   with the note that
   $$\lim(1 + \frac{1}{m})^m = e.$$

   (c) For any small fixed $\epsilon > 0$, show that the series $\sum_{k=0}^{\infty} x^k$ converges uniformly to $s(x)$ on $[-1 + \epsilon, 1 - \epsilon]$. Hint: show that $s_m(x) \rightrightarrows s(x)$ on this interval as a sequence of functions.

2. Consider the series of functions
   $$\sum_{n=1}^{\infty} \frac{x^k}{k}.$$

   (a) For any small $\epsilon > 0$, show that
   $$\left(\sum_{n=1}^{\infty} \frac{x^k}{k} \right)' = \frac{1}{1 - x}$$
   where the convergence is uniform on $[-1 + \epsilon, 1 - \epsilon]$. Hint: derivative of a sum is the sum of derivatives with proper conditions.

   (b) Show that the convergence in part (a) is pointwise on $(-1, 1)$. Hint: for a pointwise convergence, we fixed an $x_0 \in (-1, 1)$ and part (a) is true for any small ϵ. See the idea of extending the differentiability on interval $[a, b]$ to $\mathbb{R}$ at the end of the lecture.