Problems for lecture 33

April 13, 2015

This homework set is about the summation by parts formula

$$\sum_{k=m+1}^{n} a_k.b_k = s_n b_{n+1} - s_m b_{m+1} + \sum_{k=m+1}^{n} s_k(b_k - b_{k+1})$$

where $s_n = a_0 + a_1 + a_2 + \cdots + a_n$. You can see this formula in Exercise 2.7.12 on page 68 in the book. In fact, in the definition of s_n we can start at any number (for example we can define $s_n = a_3 + \cdots + a_n$ and the formula still works).

1. Assume that $\sum_{k=0}^{n} k = 0 + 1 + 2 + 3 + \cdots + n = n(n+1)/2$. Use the summation by parts formula to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. All the tests for convergence techniques for series (integral test, comparison test, p-test...) we studied in Calc. II do not work well with the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}.$$

We will show that this series converges by the steps below:

(a) Use the trigonometric identity

$$\sin m \sin \frac{1}{2} = \frac{\cos(m - 1/2) - \cos(m + 1/2)}{2}$$

to show that

$$\sin 1 + \sin 2 + \sin 3 + \cdots + \sin n = \frac{\cos(1/2) - \cos(n + 1/2)}{2\sin(1/2)}.$$

(b) If we let the summation in part (a) be s_n. It is not hard to see that $|s_n| \leq 1/\sin(1/2)$ (do explain why). Apply the summation by parts to show the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

converges. Hint: apply the formula to the sum $\sum_{k=1}^{n} \frac{\sin k}{k}$ where $n \to \infty$ and note that the series

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

is convergent by the telescoping series.