Problems for lecture 36

April 20, 2015

1. Recall that if the series $\sum_{n=0}^{\infty} a_n x^n$ converges pointwise on $(-R, R)$ then the series of derivatives $\sum_{n=1}^{\infty} n a_n x^{n-1}$ also converges pointwise on $(-R, R)$. As a consequence of Problem 1c Lecture 34, the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly on any compact subset of $(-R, R)$. Assume that $\sum_{n=0}^{\infty} a_n x^n$ converges pointwise on $(-R, R)$ to a function $f(x)$, i.e. $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where the convergence is pointwise on $(-R, R)$.

   (a) Show that at any $x \in (-R, R)$

   $$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

   (i.e., show that the convergence in this equation is pointwise on $(-R, R)$). It is equivalent to say that at any fixed x, we have $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Hint: Given a fixed x, check the conditions for the equation above on an appropriate interval.

   (b) Argue that for any fixed k, we can compute the k-th derivative of $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ at any x on $(-R, R)$. As a consequence, show that

   $$f^{(k)}(0) = k! a_k.$$

   Hint: argue that we can apply part (a) to the series of $f'(x)$, $f''(x)$,....

   (c) Show that if both series $\sum_{k=0}^{\infty} a_k x^k$ and $\sum_{k=0}^{\infty} b_k x^k$ converge pointwise to the same function $f(x)$ on $(-R, R)$ then it must be the case that $a_k = b_k$ for all k. Hint: look at part (b).