Problems for Lecture 5

1. We consider the sequence defined recursively by $x_1 = 3$ and
$$x_{n+1} = 4 - \frac{1}{x_n}.$$
We have showed in the previous homework that $3 \leq x_n \leq 2 + \sqrt{3}$ for all n. You can use this fact in this problem.
   (a) Prove that the sequence is increasing by showing that
$$4 - \frac{1}{x_n} \geq x_n$$
for all n.
   (b) The Monotone Convergence Theorem says the sequence will converge. Thus there is an s such that $s = \lim x_n$. Taking the limit of both sides of
$$x_{n+1} = 4 - \frac{1}{x_n}$$
and note that $\lim x_{n+1} = \lim x_n$ to find the limit s.

2. We consider the sequence of closed nested intervals
$$\begin{aligned} I_1 &= [a_1, b_1] \\ I_2 &= [a_2, b_2] \\ I_3 &= [a_3, b_3] \\ &\vdots \end{aligned}$$
where $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. In the lecture we show that $\sup\{a_1, a_2, \ldots\}$ belongs to I_n $\forall n \in \mathbb{N}$. Use a similar argument to show that $\inf\{b_1, b_2, \ldots\}$ also belongs to I_n $\forall n \in \mathbb{N}$.

3. Recall that for a sequence (a_n), we can form a subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ where $1 \leq n_1 < n_2 < n_3 < n_4 < \ldots$, $n_k \in \mathbb{N}$.
   (a) Prove by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.
   (b) Show that if $\lim a_n = a$ then $\lim a_{n_k} = a$ for any subsequence a_{n_k}. Hint: We need to show for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that if $k \geq N$ then
$$|a_{n_k} - a| < \epsilon.$$

(c) A consequence of part (a) is that the sequence (a_n) is NOT convergent if we can find two subsequences that are convergent to different values. Use this to show why the sequence $(-1)^n$ is not convergent.