Problems for Lecture 9

February 4, 2015

1. Consider the series $\sum a_n$.

   (a) Show that if the series $\sum a_n$ converges absolutely then the series $\sum a_n^2$ converges absolutely. Hint: if we have two positive numbers a, b then $a^2 + b^2 \leq (a+b)^2$. This is still true if we have more than two positive numbers. Do explain why.

   (b) The conclusion in part (a) may not be true if the series $\sum a_n$ simply converges (without the absolute condition). Find an example (without a formal proof) of a convergent series $\sum a_n$ where $\sum a_n^2$ diverges.

   (c) Find an example (without a formal proof) of a convergent series$\sum a_n$ and $a_n \geq 0$, where the series $\sum \sqrt{a_n}$ diverges.

2. The goal of this problem is to prove a part of the Limit Comparison Test. Let a_n and b_n be two sequences which satisfy $a_n, b_n > 0$, for all n, and

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

   where $0 < c < \infty$. Assume that the series $\sum a_n$ converges.

   (a) Using the definition of $\lim \frac{a_n}{b_n} = c$ to show that we can find an N so that if $n \geq N$ then $b_n < 2a_n/c$. Hint: choose $\epsilon = c/2$ in the definition of $\lim \frac{a_n}{b_n} = c$ and use triangle inequality.

   (b) Show that the series $\sum b_n$ converges. Hint: Use the Cauchy condition together with part (a).

3. Let a_n be a nonnegative sequence of numbers. Show that the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the sequence of partial sum

   $$s_n = a_1 + a_2 + \cdots + a_n$$

   is bounded. Hint: by definition $\sum_{k=1}^{\infty} a_k$ is convergent if and only the sequence s_n is convergent.