Problems for lecture 24

March 20, 2015

1. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Prove that there is a point $c \in (a, b)$ where
$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

   Hint: Apply the Mean Value Theorem to the function
$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x).$$

2. Assume f and g are continuous functions defined on an open interval I containing a point d, and assume that f and g are differentiable on $I \backslash \{d\}$ and $g'(x) \neq 0$ on $I \backslash \{d\}$. Prove that if $f(d) = 0$, $g(d) = 0$, and
$$\lim_{x \to d} \frac{f'(x)}{g'(x)} = L$$
   then
$$\lim_{x \to d} \frac{f(x)}{g(x)} = L.$$

   Hint: Let $x_n \to d$, $x_n \neq d$. What do we need to show? Apply Problem 1 with appropriate a, b to get
$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$
   In your argument, be careful not to divide by 0.

3. Recall that a function $f : (a, b) \to \mathbb{R}$ is increasing on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). Assume that f is differentiable on (a, b). Show that f is increasing on (a, b) IF AND ONLY IF $f'(x) \geq 0$ for all $x \in (a, b)$. Hint: Use the mean value theorem to prove that the function is increasing. Do check to make sure that you have all the conditions required for the mean value theorem. Use the limit definition of derivative to prove $f'(c) \geq 0$ at any point $c \in (a, b)$.